Post-Filtering Output Feedback Variable Structure Control

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Abstract— A full state constructed sliding mode is popularly used in a variable structure control system. To encounter with the insufficient measurement problem of the system state, a post-filtering approach to variable structure output feedback control is proposed. The proposed post-filter is used to convert the linear time invariant SISO system output into a pseudo-state sliding mode dynamics which is suitable for conventional VSS control law. It is a dynamic variable structure output feedback approach without a state observer. Stability of the digital post-filter is studied judiciously. The noncausal problem, due to the system with relative degree higher than one, discretization for inverse transfer function of the post-filter is recommended and concluded with the stable zeros conditions proposed by Aström et al.

I. INTRODUCTION

The state feedback control concept is widely used in various control theory. In the variable structure control (VSC), a sliding mode hyperplane is set up deliberately such that perfect rejection of the matching disturbance is amazingly achieved [1]. However, the requirement of full state accessibility constrained the design engineers to realize the variable structure system (VSS). The extra sensor modules need be arranged in order to avoid the unexpected signal interference. This complicates the design problem and also brings up the overall system cost. The state estimator (or the state observer) first proposed by Luenberger is the most direct solution [2]-[3]. The unavailable system state is substituted with the approximated one. After that the various observer-based VSC controllers have still been developed. Since the system output can be more easily and economically obtained than the system state in most physical systems, the system output-based VSC approaches also attracted lots of interest in the past few decades.

If the system state is neither measurable nor accessible, the observer-type VSS approach provides the most general scheme to solve this problem [4]-[5]. Linear or sliding mode observers are popularly used in conjunction with conventional VSC to serve the control purposes. Zhu et al. developed the combined algorithm for robot control. The interaction of the controller and the observer is investigated [6]. Rundell et al. presented the combined approach for stabilization of the rotational motion of a vertical shaft magnetic bearing. The sliding mode observer is used to estimate the system state and the disturbance [7]. Korondi et al. introduced an observer-based discrete-time sliding mode controller for the two-mass system coupled by a flexible shaft [8]. Ahmed-Ali et al. combined the sliding observer with a sliding mode via the backstepping procedure. The output tracking for a class of uncertain triangular nonlinear systems is achieved [9]. Xiong et al. proposed the sliding mode observer for nonlinear uncertain systems with much less conservative conditions. They rendered the nonlinear system into two subsystems and then used the equivalent control concept based on this coordinate transformation [10].

Since the overall dynamic system becomes more complex while the auxiliary state observer is involved, the system output-based VSC approaches sprang up. Some of them sought to construct a static sliding surface in the range of the output subspace [11]-[13]. The remarkable advantage is to keep off the state observer in the control loop. This concise control framework is very contributive to realization. Nevertheless, this leads to an output feedback eigenstructure assignment problem with reduced order dynamics. To carry out the design procedure, the dynamic system must satisfy the well-known ‘Kimura-Davidson’ conditions [12]. The dynamic compensator of appropriate size is then developed without the above restriction [14]-[15]. A dynamic output feedback control generally has better performance than that of static feedback but the design procedure is more complicated. In [14], both the compensator type and the observer type of controllers are proposed by Dong et al. for stabilizing the multivariable linear systems. The major difference is that the observer type introduces the control as an input to the controller, while the observer type does not. Furthermore, the compensator type controller requires a minimum phase condition, while the observer type does not [14]. Edwards et al. presented the dynamic compensator based output feedback sliding mode controller that need not
satisfy the ‘Kimura-Davidson’ conditions [15]. Saaj et al. combined the sliding controller with the fast output sampling feedback technique for single-input-single-output (SISO) system. By proper design in the reaching law and the fast output sampling feedback technique, this methodology is more practical and easy for realization [16]. It can be considered as an improved version for the static output feedback approach. Ha et al. presented the dynamic output feedback sliding mode controller based on the pole placement technique [17]. A linear functional observer is also developed to implement the sliding function and the equivalent control. In this paper, a novel variable structure output feedback approach with a post-filter is proposed. Figure 1 shows the closed-loop system functional block diagram which is similar to the idea summarized by Young et al. [18], but the broad meaning of the called ‘filtering’ function we used here can be whatever type of control process.

Figure 1. The functional block diagram of proposed closed-loop system

In our approach, the sliding vector is designed first as if the system state is fully accessible. The full state-sliding vector is a baseline and can be easily constructed according to the VSC theory. The proposed post-filter \( P(s) \) is used to convert the linear time-invariant SISO system output into the pseudo state sliding mode dynamics. The synthesized sliding mode dynamics is then introduced into the conventional variable structure control law instead of the full state one.

The other parts of this paper are organized as following. Section II represents the design concept to synthesis the sliding mode dynamics briefly. Section III discusses both the digital realization problem and the stability issue. The numerical simulation examples are given in section IV. Final section is the conclusion.

II. THE POST-FILTER

Consider a linear time invariant (LTI) SISO system with relative degree one

\[
\dot{x}(t) = A x(t), \quad i = 1, \ldots, n-1.
\]

\[
\dot{x}_n(t) = -\delta_n x_n(t) + u(t) + v(t), \quad (1a)
\]

\[
y(t) = \sum_{j=1}^{n-1} c_{j+1} x_j(t), \quad (1b)
\]

where \( x_i(t), \ldots, x_n(t) \) are the state variables, \(-\delta_1, \ldots, -\delta_n\) are the relative coefficients, \( u(t) \) is the control input, \( y(t) \) is the system output, \( c_0, \ldots, c_{n-1} \) are the relative output coefficients, and the unknown disturbance \( v(t) \) is a smooth function of time with known bound

\[|v(t)| < v_{\text{max}}. \quad (1c)\]

Let \( s = \frac{d}{dt} \) denote the differential operator. Using the Laplace transformation, the transfer function of the above LTI SISO system is shown in Figure 2.

![Figure 2. The transfer function of the LTI SISO system](image)

If the system state is neither measurable nor accessible, a post-filter is introduced to construct a pseudo-state sliding surface from the system output. The synthesized one, instead of using the originally designed full-state sliding surface, is still complied with the conventional VSC theory as if the system state is fully accessible. At the beginning, the full state-sliding vector is designed deliberately in order to acquire desired performance and becomes a baseline for the constructed pseudo-state one. According to the VSC theory, the prescribed full-state sliding surface \( \Sigma(x) \) is

\[\Sigma(x) = \Lambda x = [\lambda_0, \ldots, \lambda_{n-2}] \] \[x = 0, \quad (2)\]

where \( x \in \mathbb{R}^n \) is the state vector, and \( \Lambda \in \mathbb{R}^{n \times n} \) is chosen properly. The relationship of the sliding surface and the system output can be found as

\[c_n \Sigma^{(n-1)}(x) + c_{n-2} \Sigma^{(n-2)}(x) + \cdots + c_1 \Sigma^1(x) + c_0 \Sigma(x) = y^{(n-1)} + \lambda_{n-2} y^{(n-2)}(x) + \cdots + \lambda_1 y(1) + \lambda_0 y(x). \quad (3)\]

**proof**

The dynamic equation (1) can be written as following

\[x_1^{(0)} = x_1, \quad x_1^{(1)} = x_2, \quad x_1^{(2)} = x_3, \quad \ldots, \quad x_1^{(n-1)} = x_n, \quad x_1^{(n)} = x_n, \quad (4)\]

The full-state sliding surface (2) becomes

\[\Sigma(x) = x_n + \lambda_{n-2} x_{n-2} + \cdots + \lambda_2 x_2 + \lambda_1 x_1 + \lambda_0 x_0. \quad (5)\]

Taking differentiation of (5) with respect to time in sequence,
we obtain
\[ \Sigma^{(1)}(x) = x_n^{(1)} + \lambda_{n-1}x_{n-1}^{(1)} + \cdots + \lambda_1x_1^{(1)} + \lambda_0x_1^{(1)} \]
\[ = x_n^{(2)} + \lambda_{n-2}x_{n-2}^{(2)} + \cdots + \lambda_2x_2^{(2)} + \lambda_1x_1^{(2)} \]
\[ = x_2^{(n-1)} + \lambda_{n-2}x_2^{(n-2)} + \cdots + \lambda_2x_2^{(2)} + \lambda_1x_2^{(1)} + \lambda_0x_2, \quad (6) \]

\[ \Sigma^{(j-1)}(x) = x_n^{(j-1)} + \lambda_{n-2}x_{n-2}^{(j-1)} + \cdots + \lambda_jx_j^{(j-1)} + \lambda_{j-1}x_{j-1}^{(j-1)} \]
\[ = x_n^{(j)} + \lambda_{n-2}x_{n-2}^{(j)} + \cdots + \lambda_jx_j^{(j)} + \lambda_{j-1}x_{j-1}^{(j)}, \quad (7) \]

\[ \Sigma^{(n-1)}(x) = x_n^{(n-1)} + \lambda_{n-2}x_{n-2}^{(n-1)} + \cdots + \lambda_1x_1^{(n-1)} + \lambda_0x_1^{(n-1)} \]
\[ = x_n^{(0)} + \lambda_{n-2}x_{n-2}^{(0)} + \cdots + \lambda_1x_1^{(0)} + \lambda_0x_1, \quad (8) \]

The above differential equations can be summarized as
\[ \Sigma^{(j)}(x) = x_n^{(j)} + \lambda_{n-2}x_{n-2}^{(j)} + \lambda_{n-3}x_{n-3}^{(j)} + \cdots + \lambda_jx_j^{(j)} + \lambda_{j-1}x_{j-1}^{(j)}, \quad (9) \]

To find out the relationship between the sliding surface and the system output, we define the auxiliary equation as
\[ \eta(x) = [c_0 \quad c_1 \quad \cdots \quad c_{n-2} \quad c_{n-1}]. \quad (10) \]

To introduce (9) into (10), the auxiliary equation becomes
\[ \eta(x) = c_n \Sigma^{(-1)}(x) + c_{n-2} \Sigma^{(n-2)}(x) + \cdots + c_1 \Sigma^{(1)}(x) + c_0 \Sigma(x) \]
\[ = x_n^{(n-1)} + \lambda_{n-2}x_{n-2}^{(n-1)} + \cdots + \lambda_1x_1^{(n-1)} + \lambda_0x_1 \]
\[ + c_{n-2}x_{n-2}^{(n-2)} + \cdots + c_1x_1^{(n-2)} + c_0x_1 \]
\[ + \lambda_{n-2}x_{n-2}^{(n-2)} + \cdots + \lambda_1x_1^{(n-2)} + \lambda_0x_1 \]
\[ + \cdots + c_{n-2}x_{n-2}^{(0)} + c_1x_1^{(0)} + c_0x_1 \]
\[ + \lambda_{n-2}x_{n-2}^{(0)} + \cdots + \lambda_1x_1^{(0)} + \lambda_0x_1 \] \quad (11)

Therefore,
\[ \eta(x) = [c_0 \quad c_1 \quad \cdots \quad c_{n-2} \quad c_{n-1}]. \quad (10) \]

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\[ \eta(x) = c_n \Sigma^{(-1)}(x) + c_{n-2} \Sigma^{(n-2)}(x) + \cdots + c_1 \Sigma^{(1)}(x) + c_0 \Sigma(x) \]
\[ = x_n^{(n-1)} + \lambda_{n-2}x_{n-2}^{(n-1)} + \cdots + \lambda_1x_1^{(n-1)} + \lambda_0x_1 \]
\[ + c_{n-2}x_{n-2}^{(n-2)} + \cdots + c_1x_1^{(n-2)} + c_0x_1 \]
\[ + \lambda_{n-2}x_{n-2}^{(n-2)} + \cdots + \lambda_1x_1^{(n-2)} + \lambda_0x_1 \]
\[ + \cdots + c_{n-2}x_{n-2}^{(0)} + c_1x_1^{(0)} + c_0x_1 \]
\[ + \lambda_{n-2}x_{n-2}^{(0)} + \cdots + \lambda_1x_1^{(0)} + \lambda_0x_1 \] \quad (11)

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\[ + c_{n-2}x_{n-2}^{(n-2)} + \cdots + c_1x_1^{(n-2)} + c_0x_1 \]
\[ + \lambda_{n-2}x_{n-2}^{(n-2)} + \cdots + \lambda_1x_1^{(n-2)} + \lambda_0x_1 \]
\[ + \cdots + c_{n-2}x_{n-2}^{(0)} + c_1x_1^{(0)} + c_0x_1 \]
\[ + \lambda_{n-2}x_{n-2}^{(0)} + \cdots + \lambda_1x_1^{(0)} + \lambda_0x_1 \]

III. DIGITAL REALIZATION OF THE POST-FILTER \( P(s) \)

Due to the revolutionized diffusion of digital controllers, discrete time implementations of variable structure systems become one of the main topics in modern VSC theory. Properness and stability of the proposed post-filter are both of the major concerns in digital realization. If \( P(s) \) is proper, it can be discretized directly for digital implementation. Otherwise, we encounter a realization problem with the relative degree higher than one. The realization approaches and the stability issue are presented and discussed in this section.

A. Discretization

(i) If the coefficient \( c_{n-1} \neq 0 \), \( P(s) \) is obviously proper. To discretize (13) by zero-order-hold process yields the general form
\[ P(z) = \Sigma(z) Y(z) = z^{a_0} + z^{a_{-1}} Y(z) \]
\[ = a_1z^{a_2} + \cdots + a_{n-2}z^{a_{n-2}} + a_{n-1}z^{a_{n-1}} \]

where \( b_0, \ldots, b_{n-1}, a_1, \ldots, a_{n-1} \) are suitable coefficients. The discretized post-filter becomes the \((n-1)^{th}\)-order general form of the infinite-impulse-response (IIR) filter
\[ P(z^{-1}) = \Sigma(z^{-1}) Y(z^{-1}) = \frac{b_0 + b_1 z^{-1} + \cdots + b_{n-1} z^{-(n-1)}}{1 + a_1 z^{-1} + \cdots + a_{n-2} z^{-(n-2)} + a_{n-1} z^{-(n-1)}} \]
where \( a_0 = 1 \). Therefore, the discrete-time sliding mode \( \Sigma_k \) now becomes

\[
\Sigma_k = \sum_{i=0}^{r-1} b_i y_{k-i} + \sum_{i=0}^{r-1} a_i z_i - a_0 \Sigma_{k-1} - \cdots - a_{r-1} \Sigma_{k-r+1} - a_0 \Sigma_{k-r+1} - \cdots - a_{r-1} \Sigma_{k-r+1}.
\]

(ii) If \( c_{n-1} = 0 \) and \( c_{n-2} \neq 0 \), the inverse of \( P(s) \) is discretized instead of \( P(s) \) and yields

\[
\frac{1}{P(z)} = \frac{a_0 z^{-2} + a_1 z^{-3} + \cdots + a_{r-2} z + a_{r-1}}{z^{-1} + \beta_1 z^{-2} + \cdots + \beta_{r-1} z + \beta_0},
\]

where \( a_0, a_1, \ldots, a_{r-1}, \beta_1, \beta_2, \ldots, \beta_{r-1} \) are suitable coefficients. The difference equation becomes

\[
y_{k+1} + \beta_1 y_{k+2} + \cdots + \beta_{r-1} y_{k+r-2} + \alpha_0 \sum_{i=0}^{r-2} y_{k-i} + a_0 \Sigma_{k-1} + a_0 \Sigma_{k-2}.
\]

If \( a_0 \neq 0 \), the discrete-time sliding mode \( \Sigma_k \) can be expressed as

\[
\Sigma_k = \alpha_0 \left( y_{k+1} + \beta_1 y_{k+2} + \cdots + \beta_{r-1} y_{k+r-2} - \alpha_0 \Sigma_{k-1} - \cdots - a_{r-1} \Sigma_{k-r+1} - a_0 \Sigma_{k-r+1} \right).
\]

However, the above equation is not causal since the future output value \( y_{k+1} \) is involved at the \( k \)th sampling instant. If the system is designed to be BIBO stable, a similar idea used in [19] is introduced to approximate \( \Sigma_k \) with \( \Sigma_{k-1} \). Hence,

\[
\Sigma_{k-1} = \alpha_0 \left( y_{k+1} + \beta_1 y_{k+2} + \cdots + \beta_{r-1} y_{k+r-2} - \alpha_0 \Sigma_{k-1} - \cdots - a_{r-1} \Sigma_{k-r+1} - a_0 \Sigma_{k-r+1} \right).
\]

Remark 1. The improper post-filter is due to the physical system with relative degree two or higher. Hence, the extra derivative of the system output is unavoidable. Fridman pointed out that any differentiation would involve some parasitic dynamics that would inevitably result in loss of accuracy and chattering [21]. Although the discretized problem of the improper \( P(s) \) can be solved by Corollary 1, the seriously degraded system performance should be evaluated carefully in practical applications.

### IV. Numerical Example

Let \( \Sigma(x) = Ax = [a_0 \ a_1]x = [1 \ 1]x \). If a given dynamic system is full state accessible, the most conventional and extensively used discrete-time VSS control is

\[
u_i = -K \cdot \text{sgn}(\Sigma_k),
\]

where \( K \) is the proper gain value. This switching type VSS control (22) will be used as a baseline to evaluate the closed-loop performance with the proposed post-filter.

A. \( P(s) \) is proper

Given the linear time-invariant SISO system

\[
\dot{x} = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & -3 & 3
\end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 
\end{bmatrix} v,
\]

\[
y = \begin{bmatrix} \frac{1}{2} & 1 \end{bmatrix} x,
\]

with the initial condition \( x(0) = 3, x_2(0) = -1, x_3(0) = 2 \), and the unknown external disturbance \( v(t) = 0.5 + 5 \sin(0.1t) - \cos t \).

If the system state of (23) is not accessible, the post-filter \( P(s) \) is constructed as
To discretize $P(s)$ with the sampling period $T=0.01$ second yields

$$P(z) = \frac{\Sigma(z)}{Y(z)} = \frac{z^3 - 1.9899z + 0.99}{z^3 - 1.9988z^2 + 0.9989}.$$  \hspace{2cm} (25)

From (25), the sliding mode dynamics becomes

$$\Sigma = y_4 - 1.9899y_{k-1} + 0.99y_{k-2} - 1.9988y_{k-3} - 0.9989y_{k-4}. \hspace{2cm} (26)$$

Now (26) can be simply introduced to the conventional VSS control (22). Let the gain value $K=12$. The closed-loop system performance is compared and shown in Figure 3. The comparison of the state trajectory is shown in Figure 4.

It is obviously that the proposed approach with the fictitious sliding variable, obtained from the system output via the post-filter, still has perfect rejection of the matching disturbance in the VSS.

B. A flexible system with improper $P(s)$ [22]

Let us consider a similar system model used by Meckl et. al. as follows [22]

$$\frac{Y(s)}{U(s)} = \frac{bs + k}{(m_1 + m_2)(s^4 + bs^3 + ks^2)}, \hspace{2cm} (27)$$

where $U(s)$ and $Y(s)$ represent the Laplace transform of the applied force $u$ and the endpoint position $y$ respectively. The nominal system parameters are given by $m_1=4$, $m_2=5$, $k=8$, and $b=2$. The dynamic equation becomes

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = -8x_1 - 2x_4 + u + v, \quad \dot{x}_4 = \frac{8}{9}x_1 + \frac{2}{9}x_2, \hspace{2cm} (28)$$

with the initial condition $x_1(0)=0.5$, $x_2(0)=-1$, $x_3(0)=2$, $x_4(0)=5$, and the external disturbance $v(t) = 1 + 5\sin(10t) - \cos t$.

The relative degree of the above mechanical system is 3. Let the sliding surface be $\Sigma(x) = x_4 + \lambda_2x_3 + \lambda_1x_2 + \lambda_0x_1$. According to (13), the proposed post-filter $P(s)$ is constructed as

$$P(s) = \frac{\Sigma(s)}{Y(s)} = \frac{s^3 + \lambda_2s^2 + \lambda_1s + \lambda_0}{\frac{2}{9}s^3 + \frac{8}{9}}. \hspace{2cm} (29)$$

The coefficients $\lambda_0$, $\lambda_1$, and $\lambda_2$, which determine the zeros of $P(s)$, are chosen deliberately in order to obtain both of the desired sliding mode dynamics and the stability conditions (21). Since the pole of $P(s)$ is $-4$, a reasonable choice for the zeros in (29) are $-0.5$, $-2$, and $-6$, respectively. The sliding vector becomes $A=[\lambda_0 \lambda_1 \lambda_2 1]=[6 16 8.5 1]$.

To realize $P(s)$ with (17), we obtain

$$1 - \frac{Y(z)}{P(z)} = \frac{0.1095e^{-0.044}z + 0.0027e^{-0.044}z - 0.1036e^{-0.044}}{z^2 - 2.9170z + 2.8355 \quad 0.9185}. \hspace{2cm} (30)$$

The stable zeros of $1/P(z)$ are $-0.9851$, and $0.9608$. The poles of $1/P(z)$ are $0.9950$, $0.9802$, and $0.9418$ which are all in the unit circle. From (20), the approximated sliding mode is

$$\Sigma_{k+1} = 91324y_k - 266390y_{k-1} + 258950y_{k-2} - 83881y_{k-3} - 0.0247\Sigma_{k-2} + 0.9461\Sigma_{k-3} \approx \Sigma_k. \hspace{2cm} (31)$$

Now (31) can be applied to the modified VSS control law

$$u_k = -K[\text{sgn}(\Sigma_k)]. \hspace{2cm} (32)$$
where gain value $K=50$. Figure 5 shows the performance of both (32) with improper $P(s)$ and (22) of full state accessibility.

We observe that the performance of the full state accessible VSS method is much better than the proposed post-filtering approach. This is due to the system with relative degree two or higher as the analysis by Fridman [21]. Although the proposed approach provides an alternate and simple solution for a LTI SISO system with relative degree one than ever, this limitation for real applications should be noticed.

V. CONCLUSION

In most physical systems, the system state is not always accessible. The full-state feedback VSS method has its constraint to deal such a case. In this paper, a post-filtering approach to LTI SISO system output feedback control is proposed. The major significance of the proposed post-filter is to synthesize a pseudo-state sliding surface instead of the full-state sliding surface in VSS control loop. The VSC design procedure can be easily applied with the system output as if the full system state is accessible. The post-filter requires only partial information of the system parameters, $c_i$'s. The uncertainties of $\delta_i$'s do not influence the structure of the post-filter. In fact, the variations of $\delta_i$'s still remain in the closed-loop system. However, they can be considered as the matched uncertainties for SISO system. Furthermore, since the post-filter does not need the control signal $u$ as input, all exogenous disturbances that enters through the control channels will not affect the constructed sliding mode dynamics. Robustness can be achieved via VSC. However, an improved version of the post-filtering algorithm for the system of relative degree higher than one is necessary for us to investigate in the future.

REFERENCES