Extended Carrier-Hopping Prime Codes for Optical and Wireless CDMA Systems

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Abstract—Carrier-hopping codes, a collection of two-dimensional (2D) patterns with good correlation properties, find applications in frequency-hopping wireless code-division multiple-access (CDMA) and wavelength-hopping fiber-optic CDMA systems. This kind of codes were usually designed under the assumption that one pulse per row is used in the 2D patterns and the number of available carriers is the same as code weight. While these assumptions restrict code cardinality, there are scenarios in which the number of available carriers is more than the actual number of carriers needed in the patterns. To provide flexible code design without these assumptions, this paper reports a new family of carrier-hopping codes, so-called carrier-hopping prime codes, with significantly expanded (asymptotically optimal) cardinality, while maintaining very good correlation properties. The performance of the new codes is analyzed and compared to a carrier-grouping scheme using the original carrier-hopping prime codes. In addition, the traditional chip-synchronous assumption used in performance analysis is removed, for the first time, for 2D codes with cross-correlation functions of at most one, showing improvement and better accuracy in error probability.

I. BACKGROUND AND MOTIVATION

A two-dimensional (2D) pattern (or matrix) is a basic structure that pervades the study of topics such as frequency-hopping wireless code-division multiple-access (CDMA) and wavelength-hopping fiber-optic CDMA systems [1]–[4]. Each matrix consists of an $L \times N$ 2D signature pattern of zeros and ones with code weight $w$, comprising $w$ ones in $L$ rows (related to the number of carriers) and $N$ columns (related to the number of time slots). These code matrices, here called carrier-hopping codes, with good correlation properties find applications to frequency-hopping (or wavelength-hopping) CDMA systems, in which frequency hop (or wavelength hop) takes place at every pulses (i.e., binary ones) of a code matrix [1].

Traditionally, every matrix in these carrier-hopping codes requires at most one pulse per column in order to optimize correlation functions. While it is not always necessary, another requirement is the use of one pulse per row in order to optimize code performance, based on an assumption that the number of available carriers is the same as the code weight [4]. In particular, in multiple-access applications, if a lower code weight is already good enough to achieve a target performance, the assumption means that there are more carriers available than they are actually needed in the code matrices. In other words, by efficiently utilizing the additional carriers, we can substantially improve the code cardinality.

In this paper, a new class of carrier-hopping codes are constructed by modifying the original carrier-hopping prime codes [1] without the one-pulse-per-row restriction. The new carrier-hopping prime codes possess expanded cardinality, zero autocorrelation sidelobes, and cross-correlation functions of at most one. Furthermore, the new codes nearly achieve the upper bound on code cardinality and are, thus, asymptotically optimal. Also in this paper, the performance of the extended carrier-hopping prime codes is analyzed and compared to a carrier-grouping scheme using the original carrier-hopping prime codes. With the same number of carriers, code weight, and code length, the new codes always provide a better cardinality than the grouping scheme. Under the best scenario that a central controller is used to uniformly distribute all simultaneous users over groups of carriers (i.e., wavelength or frequency) in the grouping scheme, our results show that the new codes perform as good as the grouping scheme if the traffic load is heavy. Although the grouping scheme performs better at a light traffic load, we note that the performance of the new codes is compared to the performance lower bound of the grouping scheme. In other words, the new codes will generally have better performance if the central controller is not available to the grouping scheme. Finally, the traditional chip-synchronous assumption commonly used in performance analysis in optical CDMA is removed in this paper, for the first time, for 2D codes. Our results show that error probability is improved by eliminating the chip-synchronous assumption and the improvement increases with code weight, code length, and the number of carriers. Our technique, providing more accurate performance analysis, can also be applied to any 2D codes with cross-correlation functions of at most one.

II. EXTENDED CARRIER-HOPPING PRIME CODES

For ease of representation, every matrix in a carrier-hopping code can be written as a set of $w$ ordered pairs (i.e., one ordered pair for every binary one), where an ordered pair $(\lambda_v, \lambda_h)$ records the vertical ($v$) and horizontal ($h$) displacements of a binary one from the bottom-leftmost corner of a matrix. In other words, $\lambda_v$ represents the transmitting carrier (i.e., wavelength or frequency) and $\lambda_h$ shows the time position of a binary one in the matrix.

Definition: The $(L \times N, w, \lambda_v, \lambda_h)$ carrier-hopping code, $C$, is a collection of binary $L \times N$ matrices, each of Hamming weight $w$, such that the following properties hold [1]:

- **Autocorrelation.** For any matrix $x \in C$ and integer $\tau \in [1, N - 1]$, the binary discrete 2D autocorrelation
sidelobes of \( x \) are no greater than a nonnegative integer \( \lambda_2 \), such that \( \sum_{i=0}^{L-1} \sum_{j=0}^{N-1} x_{ij}x_{ij\oplus\tau} \leq \lambda_2 \), where \( x_{ij} = \{0, 1\} \) is an element of \( x \) at the \( i \)th row and \( j \)th column and “\( \oplus \)” denotes a modulo-\( N \) addition.

**Cross-correlation.** For any two distinct matrices \( x, y \in C \) and \( y \in C \) and integer \( \tau \in [0, N-1] \), the binary discrete 2D cross-correlation function of \( x \) and \( y \) is no greater than a positive integer \( \lambda_c \), such that \( \sum_{i=0}^{L-1} \sum_{j=0}^{N-1} x_{ij}y_{ij\oplus\tau} \leq \lambda_c \), where \( y_{ij} = \{0, 1\} \) is an element of \( y \) at the \( i \)th row and \( j \)th column.

The original carrier-hopping prime codes belong to a family of \((w \times p_1p_2 \cdots p_k, w, 0, 1)\) codes with \( p_1p_2 \cdots p_k \) matrices of length \( N = p_1p_2 \cdots p_k \) and weight \( w \), where \( L = w \) carriers are used, \( p_k \geq p_{k-1} \geq \cdots \geq p_1 \) are prime numbers and \( p_1 \geq w \). If the number of available carriers \( L \) is more than the code weight \( w \), say \( L = wp' \) and \( p' \) is a prime number, a simple way to utilize the additional carriers is to separate them into \( p' \) groups of \( w \) carriers each, and then the same \( p_1p_2 \cdots p_k \) matrices are used for each group, supporting a total of \( wp_1p_2 \cdots p_k \) subscribers. (This method is similar to the use of wavelength-division multiplexing on top of optical CDMA [6]. Here, we call this method as a carrier-grouping scheme.) To achieve a larger cardinality, the extended carrier-hopping prime codes have each of the original \( p_1p_2 \cdots p_k \) code matrices taken as a seed from which \((w+1)p' \) groups of new matrices are generated. As a result, \((w+1)p_1p_2 \cdots p_k \) new code matrices of length \( p_1p_2 \cdots p_k \) and weight \( w \), out of \( wp' \) possible carriers, are generated. As shown in Lemma 1, while the general cross-correlation functions are still at most one, some of these code matrices even have zero cross-correlation values.

### A. Construction Algorithm

From [1], when \( w = p_1 \), given a positive integer \( k \) and a set of prime numbers \( p_k \geq p_{k-1} \geq \cdots \geq p_2 \geq p_1 \), code matrices, \( x_{i_1,i_2,\ldots,i_k} \), with the ordered pairs

\[
\{(0,0), (1,i_1+i_2p_1+\cdots+i_kp_1p_2\cdots p_k-1),
\begin{align*}
&(2,2\odot p_1,i_1+(2\odot p_2,i_2)p_1+\cdots
+(2\odot p_k,i_k)p_1p_2\cdots p_k-1), \\
&(p_1-1,(p_1-1)\odot p_1,i_1+((p_1-1)\odot p_2,i_2)p_1
+\cdots+((p_1-1)\odot p_k,i_k)p_1p_2\cdots p_k-1)
\end{align*}
\}
\]

form the \((p_1 \times p_1p_2 \cdots p_k)\) original carrier-hopping prime code with \( p_1p_2 \cdots p_k \) matrices of length \( p_1p_2 \cdots p_k \) and weight \( p_1 \), where “\( \odot p_j \)” denotes a modulo-\( p_j \) multiplication and \( j = \{1, 2, 3, \ldots, k\} \).

Further, given a positive integer \( w \) and a prime number \( p' \), such that \( p' \geq w \) and \( wp' \leq p_1 \), the \( w \) ordered pairs of new matrices, \( x_{i_1,i_2,\ldots,i_k,l} \) and \( x_{i_1,i_2,\ldots,i_k,l} \), are obtained by choosing the ordered pairs from the original code matrix \( x_{i_1,i_2,\ldots,i_k} \) with the first component of each ordered pair in (1) given by

\[
\{(l_1, l_1 \oplus_w l_2) + w, (l_1 \oplus_w (2 \odot_{p'} l_2)) + 2w, \ldots, (l_1 \oplus_w ((w-1) \odot_{p'} l_2)) + (w-1)w : \\
l_1 = \{0,1,\ldots,w-1\}, l_2 = \{0,1,\ldots,p'-1\} \}
\]

and

\[
\{[l_2w, l_2w+1, \ldots, l_2w+w-1] : \\
l_2 = \{0,1,\ldots,p'-1\} \}
\]

respectively, resulting in the \((wp' \times p_1p_2 \cdots p_k, w, 0, 1)\) extended carrier-hopping prime code with \((w+1)p_1p_2 \cdots p_k \) matrices of length \( p_1p_2 \cdots p_k \) and weight \( w \), out of \( wp' \) available carriers, where “\( \odot_{p'} \)” and “\( \oplus_w \)” denote a modulo-\( p' \) multiplication and a modulo-\( w \) addition, respectively.

There are \( p'p_1p_2 \cdots p_k \) subsets of \( w \) matrices each, from (2), and \( p_1p_2 \cdots p_k \) subsets of \( p' \) matrices each, from (3), such that all matrices within a subset have zero cross-correlation values. These “subset” partitions give improved performance, and, at the same time, code cardinality.

Using \( w = p' = 3 \) and \( p_1 = p_2 = 11 \) as an example, the original carrier-hopping prime code has 121 matrices, \( x_{i_1,i_2} \) (for \( i_1 \in [0, 10] \), \( i_2 \in [0, 10] \)), represented by the ordered pairs \((0,0), (1,1)+11i_2, (2,2\odot_{11} i_1+(2\odot_{11} i_2))11, \ldots, (10,10\odot_{11} i_1+(10\odot_{11} i_2))11) \), according to (1). Simply separating the \( wp' = 9 \) carriers into 3 groups, this gives a total of 363 code matrices of weight 3 and length 121. However, following the new construction, each original matrix can generate 12 new groups of matrices, resulting in totally 1452 matrices of weight 3 and length 121. Each of the new matrices, denoted as \( x_{i_1,i_2,l_1,l_2} \) and \( x_{i_1,i_2,l_1,l_2} \) (for \( i_1 \in [0, 10] \), \( i_2 \in [0, 10] \), \( l_1 \in [0, 2] \) and \( l_2 \in [0, 2] \), is represented by 3 ordered pairs chosen from the ordered pairs of the original matrix \( x_{i_1,i_2} \) with the first component given by \((1,1) + 3, (1,1) + (2\odot_{3} l_2) + 6) \) and \((3,2,3,2) + 1, 3, 2) + 2, according to (2) and (3), respectively. For example, \( x_{2,5,1,2} \) has the ordered pairs \((1,57), (3,50), (8,82)) \}; \( x_{2,5,2,2} \) has the ordered pairs \((2,114), (4,107), (6,89)) \}; \( x_{2,5,1,1} \) has the ordered pairs \((1,57), (5,43), (6,89)) \).

**Lemma 1:** The autocorrelation sidelobes of any matrix in the \((wp' \times p_1p_2 \cdots p_k, w, 0, 1)\) extended carrier-hopping prime codes are zero and the cross-correlation function between any two distinct matrices in the code set is at most one. There are \( p'p_1p_2 \cdots p_k \) subsets of \( w \) matrices each, from (2), and \( p_1p_2 \cdots p_k \) subsets of \( p' \) matrices each, from (3), such that all matrices within a subset have zero cross-correlation values.

**Proof:** See Appendix I.

### B. Cardinality

Let \( \Phi(\times N, w, \lambda_a, \lambda_c) \) be the upper bound of the cardinality of the \((\times N, w, \lambda_a, \lambda_c)\) carrier-hopping prime code. The upper bound of the code cardinality is derived by multiplying the Johnson bound for optical orthogonal codes with \( L \) available carriers and is given by [5]

\[
\Phi(\times N, w, \lambda, \lambda, \lambda) = \frac{L(NN-1)(NN-2)\cdots(NN-\lambda)}{w(w-1)\cdots(w-\lambda)}
\]
The upper bound in (4), for the \((wp' \times p_1p_2 \cdots p_k, w, 0, 1)\) extended carrier-hopping prime code with \(L = wp'\), \(N = p_1p_2 \cdots p_k\), \(\lambda_n = 0\), and \(\lambda_r = 1\), can be modified to

\[
\Phi(wp' \times p_1p_2 \cdots p_k, w, 0, 1)
\leq \Phi(wp' \times p_1p_2 \cdots p_k, w, 1, 1)
\leq \frac{wp'(wp'p_1p_2 \cdots p_k - 1)}{w(w - 1)} = p'p_1p_2 \cdots p_k + \frac{p'^2p_1p_2 \cdots p_k - p'}{w - 1} \tag{5}
\]

Compared to the actual cardinality of the \((wp' \times p_1p_2 \cdots p_k, w, 0, 1)\) extended carrier-hopping prime code (i.e., \((w + 1)p'p_1p_2 \cdots p_k\)), (5) is larger by a factor of \(1 - 1/(wp'/w + 1) + p'/(w^2 - 1) - (wp'p_1p_2 \cdots p_k)/w\). Furthermore, if \(p' = w\), the upper bound in (5) becomes

\[
\Phi(w^2 \times p_1p_2 \cdots p_k, w, 0, 1)
\leq \frac{w^2p_1p_2 \cdots p_k - w}{w - 1} \tag{6}
\]

Compared to the actual cardinality (i.e., \((w^2 + w)p_1p_2 \cdots p_k\)), (6) is larger by a factor of approximately \(1/w^2\). The factor is nearly equal to zero for a large \(w\). The \((wp' \times p_1p_2 \cdots p_k, w, 0, 1)\) extended carrier-hopping prime code is, thus, asymptotically optimal.

### III. Performance Analysis

To analyze the performance of the extended carrier-hopping prime codes, the probability to line up (or hit) with one of the pulses (i.e., a binary one) in a signature matrix with a pulse in a received matrix is needed. The new codes have improved hit probabilities because matrices contain no common carriers and, thus, have zero cross-correlation values if they all come from the same subset. To account for this property, the average probability of getting one hit is given by [1]

\[
q = \frac{w^2}{2 \cdot L \cdot N} \times F = \frac{w}{2 \cdot p' \cdot p_1p_2 \cdots p_k} \times F \tag{7}
\]

where \(F\) represents the ratio of the number of matrices contributing one hit (to the cross-correlation function) to the total number of interfering matrices, \(w\) is the code weight, \(N = p_1p_2 \cdots p_k\) is the code length, \(L = wp'\) is the number of available carriers, and the factor \(1/2\) comes from the assumption of equiprobable 0-1 data-bit transmission.

According to the subset partitions, there are \(p'p_1p_2 \cdots p_k\) subsets of \(w\) matrices each, from (2), and \(p_1p_2 \cdots p_k\) subsets of \(p'\) matrices each, from (3), such that all matrices within a subset have zero cross-correlation values. Thus,

\[
F = \sum_{\text{all subsets}} P(\text{a matrix cases one hit|the matrix is from a subset}) \cdot P(\text{the subset is chosen from the code set})
\]

\[
= \frac{\Phi - w}{\Phi - 1} \cdot \frac{w}{(w + 1)p'p_1p_2 \cdots p_k} \cdot p'p_1p_2 \cdots p_k 
\]

\[
= \frac{\Phi - w}{\Phi - 1} \cdot \frac{p'}{(w + 1)p'p_1p_2 \cdots p_k} \cdot p_1p_2 \cdots p_k 
\]

\[
= \frac{\Phi - p'}{(w + 1)p'p_1p_2 \cdots p_k} \cdot p_1p_2 \cdots p_k 
\]

\[
= \Phi - p' \tag{8}
\]

where \(\Phi = (w + 1)p'p_1p_2 \cdots p_k\) is the code cardinality and \(\Phi - 1\) represents the total number of interfering matrices.

#### A. Analysis With the Chip-synchronous Assumption

Let \(Th\) and \(K\) denote the decision threshold of the receiver and total number of simultaneous users, respectively. The error probability of the extended carrier-hopping prime codes is given by [1]

\[
P_e = \frac{1}{2} \sum_{i = Th}^{K - 1} \binom{K - 1}{i} q^i (1 - q)^{K - 1 - i} \tag{9}
\]

where \(Th\) is usually set to \(w\) for optimal detection.

For the grouping scheme of utilizing the additional carriers by separating them into \(p'\) groups of \(w\) carriers each, code matrices from different groups will not interfere with each other because they have different carriers, even though each group uses the same matrices from the \((w \times p_1p_2 \cdots p_k, w, 0, 1)\) original carrier-hopping prime code. If all these \(p'\) groups of code matrices are uniformly distributed to the \(K\) simultaneous users by a central controller, there will be at most \([K/p']\) users transmitting simultaneously in each group, where \([\cdot]\) is the ceiling function. The error probability is then given by [1]

\[
P_e = \frac{1}{2} \sum_{i = Th}^{[K/p'] - 1} \binom{[K/p'] - 1}{i} q^i (1 - q)^{[K/p'] - 1 - i} \tag{10}
\]

where \(q = w^2/(2LN) = w/(2p_1p_2 \cdots p_k)\), \(L = w\), \(N = p_1p_2 \cdots p_k\), and \(Th = w\). Without the central controller, simultaneous users may not be uniformly distributed over the \(p'\) groups and the theoretical lower bound of performance given in (10) will not be achievable at all.

The error probabilities of the \((wp' \times p_1p_2 \cdots p_k, w, 0, 1)\) extended carrier-hopping prime code and the grouping scheme using \(p'\) groups of the \((w \times p_1p_2 \cdots p_k, w, 0, 1)\) original carrier-hopping prime code versus the number of simultaneous users are plotted, based on (9) and (10), respectively, in Figure 1. In general, the error probabilities worsen as \(K\) increases, but improve as the number of carriers, code weight, or code length increases. The step-like solid curves represent the performance lower bound of the grouping scheme, which can only be achieved theoretically with the use of a central controller so that all simultaneous users are uniformly distributed over the \(p'\) groups. The curves stop at the points where the number of simultaneous users in each group is less than the code weight (i.e., \([K/p'] \leq w\)). That is, error-free transmission can theoretically be achieved as long as \([K/p'] \leq w\) because the cross-correlation functions of the original carrier-hopping prime codes are no greater than one. Note that the performance of the new codes, shown in the dashed curves, is here compared to the performance lower bound of the
grouping scheme. The new codes will generally have a better performance than the grouping scheme when the central controller is not used. The solid curves are increasingly better than the dashed curves at small \(K\) when \(p'\) increases because the grouping scheme allows the simultaneous users to be spread out into more groups. On the other hand, the solid curves converge with the dashed curves when the traffic load is heavy (i.e., \(K/p' > w\)), showing comparable performance, even comparing to the theoretical lower bound of the grouping scheme.

Note that Figure 1 does not reveal the fact that the new codes have a factor of \(w + 1\) more matrices than the grouping scheme. To have a fair comparison, both schemes should be compared under the assumption that they have the same number of carriers, code cardinality, and code length. By picking \(p'p_1p_2 \cdots p_k\) matrices out of the code set of \(\Phi = (w + 1)p'p_1p_2 \cdots p_k\) matrices such that the number of matrices with zero cross-correlation values is maximized, the performance of the new codes will be improved. In fact, the original matrices in the grouping scheme belong to a subset of the extended matrices in (3). For a given code length \(N = p_1p_2 \cdots p_k\), the grouping scheme has \(\Phi_{\text{group}} = p_{\text{group}}p_1p_2 \cdots p_k\) code matrices if the code weight is \(w_{\text{group}}\) and the number of available carriers is \(L_{\text{group}} = w_{\text{group}}p_{\text{group}}\). If the lengths and cardinalities of both schemes are assumed to be the same, then \(\Phi_{\text{group}} = \Phi\) and we have \(p'_{\text{group}} = (w + 1)p'\). Since the number of available carriers for both schemes is assumed to be the same, we also have \(w'_{\text{group}} = w_{\text{group}}p_{\text{group}} = w_{\text{group}}p'\). Hence, \(w_{\text{group}} = w'_{\text{group}}/(w + 1)p' = w/(w + 1) < 1\). Therefore, for a fair comparison, the grouping scheme can only use \(w_{\text{group}} = 1\) and the new codes certainly outperform it.

### B. Analysis Without the Chip-synchronous Assumption

Most work on the performance analyses of optical CDMA systems is based on the assumption that users in a system are frame asynchronous but chip synchronous for ease of computation. That is, the chips of different users are perfectly aligned in time, although their bit frames may not be so. The chip-synchronous assumption has been shown as an upper bound on the performance, while the chip-asynchronous assumption gives a lower bound [7], [8]. In this section, the technique reported in [1] for performance analysis of one-dimensional codes without the chip-synchronous assumption is modified for the 2D extended carrier-hopping prime codes. Our technique can also be applied to any 2D codes with cross-correlation functions of at most one (i.e., \(\lambda_c = 1\)).

Modified from [1, Chapter 3], with the chip-synchronous assumption, we need to consider \(q_i\) of the 2D code in use, where \(q_i\) is the probability of the cross-correlation function in a time slot equal to \(i\) (for \(i \in \{0,1\}\)) and \(q_0 + q_1 = 1\). For our extended carrier-hopping prime codes, we have \(q_1 < q_0\), from (7), and \(q_0 = 1 - q_1\).

Also from [1, Chapter 3], without the chip-synchronous assumption, the cross-correlation function in a time slot depends on the amount of time shifts between the two correlating matrices and can be caused by a pulse (or nothing) from the preceding time slot and a pulse (or nothing) from the present time slot. If the cross-correlation function in a time slot is plotted against time, its shape can be triangular, rectangular, or empty. Thus, we need to consider two consecutive time slots and define \(q_{i,j}\), the probability of the cross-correlation function in the preceding time slot equal to \(i\) and the cross-correlation function in the present time slot equal to \(j\), under the chip-synchronous assumption, where \(i \in [0, \lambda_c]\) and \(j \in [0, \lambda_c]\). Since the extended carrier-hopping prime codes have \(\lambda_c = 1\), we need to find \(q_{1,1}, q_{1,0}, q_{0,1}\), and \(q_{0,0}\) in the following lemma.

**Lemma 2:** For the \((wp' \times p_1p_2 \cdots p_k, w, 0, 1)\) extended carrier-hopping prime codes, the hit probabilities are approximated, for a large \(k\), as \(q_{1,1} = 0, q_{1,0} = q_{0,1} = q\), and \(q_{0,0} = 1 - 2q\).

**Proof:** See Appendix II.

From [1, (3.14)], the variance for the chip-asynchronous case, after simplification, is given by

\[
\sigma^2_{\text{asyn}} = \sum_{i=0}^{\lambda_c} \sum_{j=0}^{i-1} \left[ j^2 + (i - j) + \frac{(i - j)^2}{3} \right] (q_{i,j} + q_{j,i})
\]

\[+
\sum_{i=0}^{\lambda_c} i^2 q_{i,i} - \left( \frac{w^2}{2LN} \right)^2 = \frac{2q}{3} - q^2 \quad (11)
\]

For the chip-synchronous case, the double summation in (11) vanishes and the variance becomes

\[
\sigma^2_{\text{syn}} = \sum_{i=0}^{\lambda_c} i^2 q_{i,i} - \left( \frac{w^2}{2LN} \right)^2 = (1 - q)q \quad (12)
\]
The difference between the variances of both cases is

$$\sigma_{\text{syn}}^2 - \sigma_{\text{asyn}}^2 = \frac{\lambda_\epsilon}{2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \left[ \frac{1}{2} i^2 + \frac{1}{2} j^2 - (j^2 + (i-j)j + \frac{(i-j)^2}{3}) \right] (q_i,j + q_{j,i}) = \frac{q}{3} \quad (13)$$

For example, using the \((3 \cdot 5) \times (17 \cdot 19), 3, 0, 1\) extended carrier-hopping prime code and \(q\) from (7), the variance difference between both cases is

$$\sigma_{\text{syn}}^2 - \sigma_{\text{asyn}}^2 = \frac{w}{3 \cdot 2p'p_1p_2\cdots p_k} \cdot \frac{(\Phi - w)w + \Phi - p'}{(\Phi - 1)(w + 1)} = 0.000399748 \quad (14)$$

since \(k = 2, N = p_1p_2 = 323, w = 3, p' = 5, \) and \(\Phi = 6460.\)

In other words, the variance in the chip-synchronous case is overestimated by

$$\nu = \frac{\sigma_{\text{syn}}^2 - \sigma_{\text{asyn}}^2}{\sigma_{\text{syn}}^2} = 33.36\% \quad (15)$$

whereas the variance of the chip-asynchronous case is \(\sigma_{\text{asyn}}^2 = (1-q)q = 0.000927571\) since \(q = 0.000928433\) from (13) and (14).

The probability of error \(P_{e|G}\) of the extended carrier-hopping prime codes (or any 2D codes with \(\lambda_\epsilon = 1\)) can be obtained by applying Gaussian approximation

$$P_{e|G} = \Theta \left( \frac{-u}{\sqrt{4(K-1)\sigma^2}} \right) \quad (16)$$

where \(\Theta = (1/\sqrt{2\pi}) \int_{-\infty}^{x} \exp(-y^2/2)dy\) is the unit-normal cumulative distribution function [1]. For the chip-synchronous case, we set \(\sigma^2 = \sigma_{\text{syn}}^2 = (1-q)q\), from (12). Similarly, we set \(\sigma^2 = \sigma_{\text{asyn}}^2 = 2q/3 - q^2\), from (11), for the chip-asynchronous case. This approximation is valid for a large number of simultaneous users \(K\), where, by the Central Limit Theorem, the total interference approaches a Gaussian distribution.

In Figure 2, the error probability \(P_{e|G}\) is plotted against the number of simultaneous users \(K\) for a CDMA system with the \((wp' \times p_1p_2\cdots p_k, w, 0, 1)\) extended carrier-hopping prime code, for \(k = 2\), with and without the chip-synchronous assumption. As expected, the performance of the chip-asynchronous case is superior to the chip-synchronous case. For instance, for 100 simultaneous users with \(w = 3, p' = 5\), and \(N = 323\), \(P_{e|G} = 3.7 \times 10^{-7}\) for the chip-synchronous case and \(P_{e|G} = 6.6 \times 10^{-10}\) for the chip-asynchronous case. The chip-synchronous assumption results in overestimating the performance by over two orders of magnitude in this example. In other words, the overestimation of \(\nu = 33.36\%\), from (15), in this \((3 \cdot 5) \times (17 \cdot 19), 3, 0, 1\) extended carrier-hopping prime code translates into more than two orders of magnitude in error-probability improvement when \(K \approx 100\). Also, note that the difference in the error probability between the two cases increases with \(w, p'\), and \(N\).

IV. CONCLUSIONS

A new class of carrier-hopping prime codes, which does not have the restriction of one pulse per row in the matrices, has been constructed. The code cardinality is asymptotically optimal and substantially expanded, without sacrificing the correlation properties. The new codes will be useful in scenarios where the number of available carriers is more than the actual number of carriers needed in the matrices. With the same numbers of carriers, code weight, and code length, our results show that the extended carrier-hopping prime codes, in general, provide a better cardinality than the grouping scheme. Under the best scenario that a central controller is used to uniformly distribute all simultaneous users over the carrier groups in the grouping scheme, the new codes perform as good as the grouping scheme if the traffic load is heavy. Although the grouping scheme performs better at a light traffic load, we note that the performance of the new codes is comparing to the performance lower bound of the grouping scheme. In other words, the new codes will generally have better performance if the central controller is not available to the grouping scheme. Finally, we reported, for the first time, the performance analysis of 2D codes (with cross-correlation functions of at most one) without the traditional chip-synchronous assumption. Our results show that error probability is improved when the chips-synchronous assumption is removed and the improvement increases with code weight, code length, and the number of carriers.

APPENDIX

I. PROOF OF Lemma 1

For any two distinct new matrices \(x_{i_1,i_2,\ldots,i_k,l_1,l_2}\) and \(x'_{i_1',i_2',\ldots,i_k',l_1',l_2'}\) originated from two different matrices \(x_{i_1,i_2,\ldots,i_k}\) and \(x'_{i_1',i_2',\ldots,i_k'}\) (for \((i_1,i_2,\ldots,i_k) \neq (i_1',i_2',\ldots,i_k')\)) in the original carrier-hopping prime codes,
the cross-correlation values are at most one. It is be-

proposition only when

of the

of the

the assumption that

lengths. This gives one subset of

respectively, and the maximum cross-correlation value of

is always one [1]. The same property holds for any two distinct matrices

For any two distinct new matrices

which are originated from the same matrix

in the original code, in the

carrier-hopping prime code, the first component of each ordered pair is given by, according to (2),

and

respectively. If there exist at least two coincidences of binary ones in any relative horizontal cyclic shift of the two matrices, there must exist at least two ordered pairs with the same first component, such that

and

must hold simultaneously, where

and

Since

and

are both less than

we then have

Similarly, we also have

. Subtracting (A.4) from (A.3), we finally have

This equation is valid only when

Substituting

back into (A.3), we have

which violates the assumption that

and

are distinct. The cross-correlation value must then be at most one. Furthermore, when

either (A.3) or (A.4) is valid only when

The cross-correlation values must then be zero for a fixed

. This gives

subsets of

matrices each, from (2), for each of the

original code matrices, such that all matrices within a subset have zero cross-correlation values.

The cross-correlation value is always zero for any two distinct matrices

because all of the binary ones in both matrices are in different wave-

lengths. This gives one subset of

matrices, for (3), for each of the

original matrices.

Using a similar argument, the autocorrelation property of the

carrier-hopping prime codes is preserved and the autocorrelation sidelobes of any matrix in the code set are always zero. Q.E.D.

II. PROOF OF LEMMA 2

For any two distinct code matrices

and

of the extended carrier-hopping prime codes, the calculation of

between

and

is equivalent to count the probability of having a binary one in

adjacent (in time slot) to a binary one in

when both matrices are chip-aligned [1, Lemma 3.4]. This condition will only happen when

for

coincidences of binary ones

because all

and

are both less than

we then have

Therefore, we have

For a large

we have

and finally

Q.E.D.

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